The Log Entropy Functional Along The Ricci Flow

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1 Introduction

In [P], Perelman introduced the entropy functional $\mathcal{W}(g, f, \tau)$ and established its monotonicity along the Ricci flow. An important application of Perelman's entropy monotonicity is for establishing the κ-noncollpasing property of the Ricci flow under a finite upper bound for the time, see [P] and [Y1]. In [Y2],[Y3] and [Y4], logarithmic Sobolev inequalities along the Ricci flow in all dimensions $n \geq 2$ were obtained using Perelman's entropy monotonicity, which lead to Sobolev inequalities and κ-noncollpasing estimates. In particular, uniform estimates without any condition on the time were obtained under the assumption that the first eigenvalue of the operator $-\Delta + \frac{R}{4}$ is nonnegative at the start of the Ricci flow.

In this paper we introduce a new entropy functional which we call the *log entropy* because of the appearance of an additional logarithmic operation compared with Perelman's entropy functional. Unlike Perelman's entropy functional, the parameter τ does not appear in the log entropy. (A term involving the time t enters into the formula of the adjusted log entropy. However, its role is very different from that played by τ in Perelman's entropy.) On the other hand, the form of the log entropy is directly related to the log gradient version of the logarithmic Sobolev inequality, see [Y2, Appendix A]. Based on Perelman's entropy monotonicity we'll establish the monotonicity of the log entropy (or the adjusted log entropy). Because of the close relation between the log entropy and the log gradient version of the logarithmic Sobolev inequality, we can then show that the log gradient version of the logarithmic

Sobolev inequality improves along the Ricci flow. (This also leads to an improvement of the logarithmic Sobolev inequalities along the Ricci flow obtained in [Y2], [Y3] and [Y4].) The Sobolev inequalities also improve along the Ricci flow in an indirect way because they follow from the logarithmic Sobolev inequality.

2 The log entropy functional

Let M be a closed manifold of dimension $n \geq 1$.

Definition 1 We define the *log entropy* functional as follows

$$\mathcal{Y}_0(g,u) = -\int_M u^2 \ln u^2 dvol + \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol \right), \tag{2.1}$$

where g is a smooth metric on M and $u \in W^{1,2}(M)$ satisfies

$$\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol > 0. \tag{2.2}$$

Here, all geometric quantities are associated with g.

More generally, we define the *log entropy* functional with *remainder* a as follows

$$\mathcal{Y}_a(g,u) = -\int_M u^2 \ln u^2 dvol + \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + a \right). \tag{2.3}$$

Next we introduce the *adjusted log entropy* which depends on an additional parameter t.

Definition 2 We define the *adjusted log entropy* with remainder a as follows

$$\mathcal{Y}_a(g,u,t) = -\int_M u^2 \ln u^2 dvol + \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + a \right) + 4at. \tag{2.4}$$

Obviously, $\mathcal{Y}_a(g,u) = \mathcal{Y}_a(g,u,0)$.

We'll need the following notation which is used in [Y2]. For a metric g on M, let $\lambda_0(g)$ denote the first eigenvalue of the operator $-\Delta + \frac{R}{4}$ for g.

3 Monotonicity of the log entropy and the logarithmic Sobolev constant

Now we consider a smooth solution $g = g(t)$ of the Ricci flow on $M \times [\alpha, T)$ for some $\alpha < T$, where α is finite. Let $u = u(t)$ be a smooth positive solution of the backward

evolution equation

$$\frac{\partial u}{\partial t} = -\Delta u - \frac{|\nabla u|^2}{u} + \frac{R}{2}u \tag{3.1}$$

such that the normalization condition

$$\int_M u^2 dvol = 1 \tag{3.2}$$

holds true for all t. Equivalently, $v = v(t) \equiv u^2(t)$ is a smooth positive solution of the conjugate heat equation of Perelman

$$\frac{\partial v}{\partial t} = -\Delta v + Rv \tag{3.3}$$

such that

$$\int_M v dvol = 1 \tag{3.4}$$

for all t. Since

$$\frac{d}{dt}\int_M u^2 dvol = \frac{d}{dt}\int_M v dvol = 0, \tag{3.5}$$

(3.2) (or (3.4)) holds true for all t iff it holds true for any one value of t.

Theorem 3.1 *Assume that $a > -\lambda_0(g(\alpha))$. Then $\mathcal{Y}_a(t) \equiv \mathcal{Y}_a(g(t), u(t), t)$ is nondecreasing. Indeed, we have*

$$\begin{aligned} \frac{d}{dt}\mathcal{Y}_a &\geq \frac{n}{4\omega}\int_M |Ric - 2\frac{\nabla^2 u}{u} + 2\frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\omega}{n}g|^2 u^2 dvol \\ &= \frac{n}{4\omega}\int_M |Ric + \nabla^2 f - \frac{4\omega}{n}g|^2 e^{-f} dvol, \end{aligned} \tag{3.6}$$

where $u = e^{-f/2}$ and

$$\omega = \omega(t) = a + \int_M (|\nabla u|^2 + \frac{R}{4}u^2) dvol|_t, \tag{3.7}$$

which is positive.

Note that if $\mathcal{Y}(t_2) = \mathcal{Y}(t_1)$ for some $t_2 > t_1$, then the above monotonicity inequality (or (4.19)) implies that

$$Ric + \nabla^2 f - \frac{1}{2(t_2 - t + \sigma)}g = 0 \tag{3.8}$$

on $[t_1, t_2]$ (and hence on $[t_1, T]$), i.e. g is a gradient shrinking soliton, where $\sigma = \frac{n}{8\omega(t_2)}$. (Note that the f here is different from the f employed in the proof of Theorem 3.1 given below. This f is used for the purpose of simplifying the expressions in the above formulas.)

Next we define for each $a > -\lambda_0(g)$ the *logarithmic Sobolev constant with the a-adjusted scalar curvature potential*

$$
\begin{aligned}
C_{S,log,a}(M, g) &= \inf\{-\int_M u^2 \ln u^2 dvol + \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + (\frac{R}{4} + a)u^2) dvol \right) : \\
&\qquad u \in W^{1,2}(M), \int_M u^2 dvol = 1\}. \qquad (3.9)
\end{aligned}
$$

In other words, $C_{S,log,a}(M, g)$ is the optimal constant (i.e. the maximal possible constant) such that the logarithmic Sobolev inequality

$$
\int_M u^2 \ln u^2 dvol \leq \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + a \right) - C_{S,log,a}(M, g) \qquad (3.10)
$$

holds true for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$. A natural question is how to estimate $C_{S,log,a}(M, g)$ for a given (M, g). The next proposition provides such an estimate in terms of the modified Sobolev constant $\tilde{C}_S(M, g)$, whose definition can be found in [Y2]. A similar estimate for $n = 2$ can be proved in the same way by using the results in [Y3]. These estimates can be applied to deal with the initial metric when we study the Ricci flow.

Proposition 3.2 *Assume $n \geq 3$. Let g be a metric on M and $a > -\lambda_0(g)$. If $a \geq -\frac{\min R^-}{4} + \tilde{C}_S(M, g)^{-2} vol_g(M)^{-2/n}$, there holds*

$$
\int_M u^2 \ln u^2 dvol \leq \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + a \right) + \frac{n}{2} \ln(2\tilde{C}_S(M, g)^2) \qquad (3.11)
$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$. If $a < -\frac{\min R^-}{4} + \tilde{C}_S(M, g)^{-2} vol_g(M)^{-2/n}$, there holds

$$
\begin{aligned}
\int_M u^2 \ln u^2 dvol &\leq \frac{n}{2} \ln \left(\int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + a \right) + \frac{n}{2} \ln(1 + \frac{B}{a + \lambda_0(g)}) \\
&\quad + \frac{n}{2} \ln(2\tilde{C}_S(M, g)^2), \qquad (3.12)
\end{aligned}
$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where

$$
B = -a - \frac{\min R^-}{4} + \frac{1}{\tilde{C}_S(M, g)^2 vol_g(M)^{2/n}}. \qquad (3.13)
$$

In other words, we have $C_{S,log,a}(M, g) \geq -\frac{n}{2} \ln(2\tilde{C}_S(M, g))$ in the former case and $C_{S,log,a}(M, g) \geq -\frac{n}{2} \ln(1 + \frac{B}{a+\lambda_0(g)}) - \frac{n}{2} \ln(2\tilde{C}_S(M, g)^2)$ in the latter case.

Proof. We have by [Y2, Theorem 3.3]

$$\begin{aligned}\int_M u^2 \ln u^2 dvol &\le \frac{n}{2}\ln\left(\tilde{C}_S(M,g)\|\nabla u\|_2 + \frac{1}{vol_g(M)^{1/n}}\right)^2 \\ &\le \frac{n}{2}\ln(2\tilde{C}_S(M,g)^2) + \frac{n}{2}\ln\left(\int_M |\nabla u|^2 dvol + \frac{1}{C_S(M,g)^2 vol_g(M)^{2/n}}\right) \\ &\le \frac{n}{2}\ln\left(\left[\int_M (|\nabla u|^2 + \frac{R}{4}u^2) dvol + a\right] + B\right) + \frac{n}{2}\ln(2\tilde{C}_S(M,g)^2),\end{aligned} \tag{3.14}$$

where

$$B = -a - \frac{\min R^-}{4} + \frac{1}{\tilde{C}_S(M,g)^2 vol_g(M)^{2/n}}. \tag{3.15}$$

Hence we deduce in the case $B \le 0$, i.e. $a \ge -\frac{\min R^-}{4} + \tilde{C}_S(M,g)^{-2} vol_g(M)^{-2/n}$

$$\int_M u^2 \ln u^2 dvol \le \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2) dvol + a\right) + \frac{n}{2}\ln(2\tilde{C}_S(M,g)^2). \tag{3.16}$$

In the case $B > 0$, i.e. $a < -\frac{\min R^-}{4} + \tilde{C}_S(M,g)^{-2} vol_g(M)^{-2/n}$, we consider the function $y = \ln(x+B) - \ln x$ for $x \ge a + \lambda_0(g)$. Clearly, y is decreasing. Hence its maximum is $y(a+\lambda_0(g)) = \ln(a+\lambda_0(g)+B) - \ln(a+\lambda_0(g))$. It follows that $\ln(x+B) \le \ln x + \ln(a+\lambda_0(g)+B) - \ln(a+\lambda_0(g))$ for all $x \ge a+\lambda_0(g)$. Consequently, there holds

$$\begin{aligned}\int_M u^2 \ln u^2 dvol &\le \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2) dvol + a\right) + \frac{n}{2}\ln(1 + \frac{B}{a+\lambda_0(g)}) \\ &\quad + \frac{n}{2}\ln(2\tilde{C}_S(M,g)^2).\end{aligned} \tag{3.17}$$

The monotonicity of the log entropy functional given in Theorem 3.1 leads to the monotonicity of the log Sobolev constant (or the adjusted log Sobolev constant), as formulated in the next theorem.

Theorem 3.3 *The adjusted logarithmic Sobolev inequality improves along the Ricci flow. More precisely, $C_{S,log,a}(M, g(t)) + 4at$ is nondecreasing along an arbitrary smooth solution $g(t)$ of the Ricci flow on M, provided that a is greater than the negative multiple of the λ_0 of the initial metric. In particular, the logarithmic Sobolev inequality improves along the Ricci flow, i.e. $C_{S,log,0}(M, g(t))$ is nondecreasing along the Ricci flow, provided that $\lambda_0 > 0$ at the start.*

Next we handle the case $\lambda_0 = 0$ at the start.

Theorem 3.4 *Assume $\lambda_0 = 0$ at the start. Then either $g = g(t)$ is a gradient soliton, in which case the logarithmic Sobolev constant $C_{S,log,a}(M, g(t))$ is independent of t for any given $a > 0$; or $C_{S,log,0}(M, g(t))$ is nondecreasing on $[\epsilon, T)$ for each $\epsilon > 0$.*

Combining Theorem 3.3 and Theorem 3.4 with Proposition 3.2 (and the corresponding result in dimension 2) we then arrive at a log gradient version of the logarithmic Sobolev inequality along the Ricci flow, which improves the corresponding results in [Y2], [Y3] and [Y4].

4 The proofs

Now we proceed to prove Theorem 3.1, Theorem 3.3 and Theorem 3.4. We consider Perelman's entropy functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dvol. \tag{4.1}$$

Setting

$$u = \frac{e^{-f/2}}{(4\pi\tau)^{n/4}}, \tag{4.2}$$

i.e.

$$f = -\ln u^2 - \frac{n}{2}\ln\tau - \frac{n}{2}\ln(4\pi), \tag{4.3}$$

we have

$$\begin{aligned}
\mathcal{W}(g, f, \tau) &= -\int_M u^2 \ln u^2 dvol + \tau \int_M (4|\nabla u|^2 + Ru^2) dvol - \frac{n}{2}\ln\tau \\
&\quad -\frac{n}{2}\ln(4\pi) - n \\
&= -\int_M u^2 \ln u^2 dvol + \tau \left[\int_M (4|\nabla u|^2 + Ru^2) dvol + 4a\right] - \frac{n}{2}\ln\tau \\
&\quad -\frac{n}{2}\ln(4\pi) - n - 4a\tau \\
&= -\int_M u^2 \ln u^2 dvol + (4\tau)\left[\int_M (|\nabla u|^2 + \frac{R}{4}u^2) dvol + a\right] - \frac{n}{2}\ln(4\tau) \\
&\quad -\frac{n}{2}\ln\pi - n - a(4\tau)
\end{aligned} \tag{4.4}$$

for an arbitary constant a.

Lemma 4.1 *Let a metric g on M be given. Assume $a > -\lambda_0(g)$. Let $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$. Then the minimum of the function*

$$h(s) = s\left[\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right] - \frac{n}{2}\ln s \tag{4.5}$$

for $s > 0$ is given by

$$\min h = \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right) + \frac{n}{2}(1 - \ln\frac{n}{2}) \tag{4.6}$$

and is achieved at the unique minimum point

$$s = \frac{n}{2}\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right)^{-1}. \tag{4.7}$$

Proof. Set $A = \int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a$. Then $h(s) = As - \frac{n}{2}\ln s$. Since $a > -\lambda_0(g)$, we have $A > 0$. Hence we have $h(s) \to \infty$ as $s \to \infty$. We also have $h(s) \to \infty$ as $s \to 0$. Consequently, h achieves its minimum somewhere. Since $h'(s) = A - \frac{n}{2s}$, the minimum is achieved at the unique minimum point $s = \frac{n}{2A}$, and then the minimum is $h(\frac{n}{2A}) = \frac{n}{2}\ln A + \frac{n}{2}(1 - \ln\frac{n}{2})$. ∎

The following lemma is a simple corollary of the above lemma.

Lemma 4.2 *Assume $a > -\lambda_0(g)$. Then there holds for each $\tau > 0$*

$$\mathcal{W}(g, f, \tau) \geq -\int_M u^2 \ln u^2 dvol + \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right) - 4a\tau + b(n), \tag{4.8}$$

where

$$b(n) = -\frac{n}{2}\ln\pi - \frac{n}{2}(1 + \ln\frac{n}{2}). \tag{4.9}$$

Moreover, we have

$$\begin{aligned} \mathcal{W}(g, f, \frac{n}{8}\omega(g, u, a)^{-1}) &= -\int_M u^2 \ln u^2 dvol + \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right) \\ &\quad -\frac{na}{2}\omega(g, u, a)^{-1} + b(n), \end{aligned} \tag{4.10}$$

where

$$\omega(g, u, a) = \int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a \tag{4.11}$$

Now let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [\alpha, T)$ and $u = u(t)$ as postulated for Theorem 3.1. Fix $\alpha \le t_1 < t_2 < T$ and define for a given $\sigma > 0$

$$\tau = \tau(t) = t_2 - t + \sigma. \tag{4.12}$$

We define $f = f(t)$ by

$$f = -\ln u^2 - \frac{n}{2}\ln\tau - \frac{n}{2}\ln(4\pi), \tag{4.13}$$

i.e.

$$u = \frac{e^{-f/2}}{(4\pi\tau)^{n/4}}. \tag{4.14}$$

Then f satisfies the backward evolution equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}. \tag{4.15}$$

By Perelman's entropy monotonicity formula we have for $g = g(t)$, $f = f(t)$ and $\tau = \tau(t)$

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M |Ric + \nabla^2 f - \frac{1}{2\tau}g|^2 \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dvol \tag{4.16}$$

on $[t_1, t_2]$. It follows that

$$\begin{aligned} \mathcal{W}(g(t_2), f(t_2), \sigma) &= \mathcal{W}(g(t_1), f(t_1), t_2 - t_1 + \sigma) \\ &\quad +2\int_{t_1}^{t_2} \tau \int_M |Ric + \nabla^2 f - \frac{1}{2\tau}g|^2 \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dvoldt. \end{aligned} \tag{4.17}$$

Proof of Theorem 3.1

Choosing $\sigma = \frac{n}{8}\omega(g(t_2), u(t_2), a)^{-1}$ in (4.17) we deduce

$$\begin{aligned} &-\int_M u^2 \ln u^2 dvol|_{t_2} + \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right)|_{t_2} - 4a\sigma + b(n) \\ &= \mathcal{W}(g(t_1), f(t_1), t_2 - t_1 + \sigma) \\ &+2\int_{t_1}^{t_2} (t_2 - t + \sigma)\int_M |Ric + \nabla^2 f - \frac{1}{2(t_2 - t + \sigma)}g|^2 \frac{e^{-f}}{(4\pi(t_2 - t + \sigma))^{\frac{n}{2}}} dvoldt \\ &\ge -\int_M u^2 \ln u^2 dvol|_{t_1} + \frac{n}{2}\ln\left(\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol + a\right)|_{t_1} \\ &-4a(t_2 - t_1 + \sigma) + b(n) \\ &+2\int_{t_1}^{t_2} (t_2 - t + \sigma)\int_M |Ric + \nabla^2 f - \frac{1}{2(t_2 - t + \sigma)}g|^2 \frac{e^{-f}}{(4\pi(t_2 - t + \sigma))^{\frac{n}{2}}} dvoldt \end{aligned} \tag{4.18}$$

It follows that

$$
\begin{aligned}
&\mathcal{Y}_a(g(t_2),u(t_2),t_2) \ge \mathcal{Y}_a(g(t_1),u(t_1),t_1)\\
&+2\int_{t_1}^{t_2}(t_2-t+\sigma)\int_M |Ric+\nabla^2 f-\frac{1}{2(t_2-t+\sigma)}g|^2\frac{e^{-f}}{(4\pi(t_2-t+\sigma))^{\frac{n}{2}}}dvoldt,
\end{aligned}
\tag{4.19}
$$

which leads to

$$
\begin{aligned}
\frac{d}{dt}\mathcal{Y}_a(g(t),u(t),t) &\ge 2\sigma\int_M |Ric+\nabla^2 f-\frac{1}{2\sigma}g|^2\frac{e^{-f}}{(4\pi\sigma)^{\frac{n}{2}}}dvol\\
&= \frac{n}{4\omega}\int_M |Ric-2\frac{\nabla^2 u}{u}+2\frac{\nabla u\otimes\nabla u}{u^2}-\frac{4\omega}{n}g|^2u^2dvol.
\end{aligned}
$$

This implies that $\mathcal{Y}_a(g(t),u(t),t)$ is nondecreasing. ■

Remark With some more work, we can show that $\mathcal{Y}_a(g(t),u(t),t)$ is actually strictly increasing unless $a=0$.

Proof of Theorem 3.3 Consider $t_1<t_2$. Given $\epsilon>0$, we choose a positive smooth function u_2 such that $\int_M u_2^2 dvol=1$ at $t=t_2$ and $\mathcal{Y}_a(g(t_2),u_2)\le C_{S,log,a}(M,g(t_2))+\epsilon$. Then we let $u=u(t)$ be the positive smooth solution of the equation (3.1) on $[t_1,t_2]$ with $u(t_2)=u_2$. By Theorem 3.1 we then have

$$
\begin{aligned}
C_{S,log,a}(M,g(t_2))+\epsilon+4at_2 &\ge \mathcal{Y}_a(g(t_2),u(t_2))+4at_2\ge\mathcal{Y}_a(g(t_1),u(t_1))+4at_1\\
&\ge C_{S,log,a}(M,g(t_1))+4at_1.
\end{aligned}
\tag{4.20}
$$

Since ϵ is arbitary, we arrive at

$$
C_{S,log,a}(M,g(t_2))+4at_2\ge C_{S,log,a}(M,g(t_1))+4at_1. \tag{4.21}
$$

■

Proof of Theorem 3.4 This follows from the arguments in [Y4] and Theorem 3.3. ■

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